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NEW EXACT SOLUTION OF THE PROBLEM OF ROTATIONALLY SYMMETRIC COUETTE–POISEUILLE FLOW

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An exact solution is obtained for the problem of steady-state viscous incompressible flow under a pressure difference in the gap between coaxial cylinders for the case where the inner cylinder rotates at a constant angular velocity. The solution differs from the classical Couette–Poiseuille result by the presence of radial mass transfer, which provides for interaction between the poloidal and azimuthal circulations. The flow rate is found to depend linearly on the angular velocity of rotation of the inner cylinder.

Key words: *Navier–Stokes equations, exact solutions, Couette–Poiseuille flow.*

1. We consider rotationally symmetric steady-state flow of a viscous incompressible fluid in the gap between infinite coaxial cylinders. Let the inner cylinder of radius R_0 rotate at a constant angular velocity ω relative to its axis, and let the outer cylinder of radius R_1 be at rest. The difference of the average pressures in two cross sections of the gap S_1 and S_2 separated by a distance h is known. It is required to determine the hydrodynamic fields of the flow that arises under these conditions and to calculate the flow rate. We assume that the z axis of the cylindrical coordinates (r, φ, z) coincides with the common axis of the cylinders and that its origin is in the section S_1.

The fluid flow is described by the Navier–Stokes equations

$$(\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = \nabla P + \nu \Delta \boldsymbol{v}, \qquad \nabla \cdot \boldsymbol{v} = 0 \tag{1}$$

supplemented by the attachment conditions on the boundaries of the gap:

$$r = R_0\colon \quad v_r = v_z = 0, \quad v_\varphi = \omega R_0, \qquad r = R_1\colon \quad v_r = v_z = v_\varphi = 0. \tag{2}$$

Here $\boldsymbol{v} = (v_r, v_\varphi, v_z)$ is the velocity, P is the pressure normalized to constant density, and ν is the kinematic viscosity.

According to the formulation of the problem, in addition to the boundary conditions, it is necessary to specify the difference of the pressures averaged over the sections S_1 and S_2:

$$\Delta P = \frac{1}{\pi(R_1^2 - R_0^2)} \int\limits_{R_0}^{R_1} \int\limits_0^{2\pi} \Big(P\Big|_{z=0} - P\Big|_{z=h}\Big) r\, d\varphi\, dr. \tag{3}$$

Assuming that the longitudinal and circumferential velocity components are linear functions of the coordinate z, we seek a rotationally symmetric steady-state solution of problem (1)–(3) in the form [1, 2]

$$v_r = \frac{\nu}{R_1} \frac{u(x)}{\sqrt{x}}, \qquad v_\varphi = \frac{\nu}{R_1} \sqrt{\frac{2}{x}} \Big(V(x) + Zv(x)\Big), \qquad v_z = -2\, \frac{\nu}{R_1} \Big(W(x) + Zu'(x)\Big),$$
$$P = P_0 + 2\, \frac{\nu^2}{R_1^2} \Big(B(x) - 4ZF'(x) - 2Z^2 G(x)\Big). \tag{4}$$

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Here P_0 is the pressure on the wall of the inner cylinder in the section S_1, $x = (r/R_1)^2$ and $Z = z/R_1$ are new dimensionless variables, and u, v, G, W, V, and F are dimensionless functions of the argument x; the prime denotes differentiation with respect to x.

Representation of the hydrodynamic fields in the form (4) reduces the Navier–Stokes equations with conditions (2) and (3) to the following boundary-value problem for a system of ordinary differential equations for the unknowns u, v, G, W, V, and F:

$$2xu''' = 2G + (u-2)u'' - u'u', \qquad 2xv'' = uv' - vu', \qquad 4x^2G' = -v^2; \tag{5}$$

$$x = x_0\colon \quad u = u' = v = 0, \qquad x = 1\colon \quad u = u' = v = 0; \tag{6}$$

$$2xW'' = 2F' + (u-2)W' - u'W, \qquad 2xV'' = uV' - vW, \qquad 4x^2F'' = -vV; \tag{7}$$

$$x = x_0\colon \quad W = 0, \quad V = \Omega, \quad F = F_0, \qquad x = 1\colon \quad W = V = F = 0; \tag{8}$$

$$B = u' - \frac{u^2}{4x} + \frac{1}{2}\int\limits_{x_0}^{x}\Big(\frac{V(t)}{t}\Big)^2\,dt.$$

Here $x_0 = (R_0/R_1)^2$, $\Omega = \omega R_0^2/(\sqrt{2}\,\nu)$ is the dimensionless component of the axial angular momentum $\boldsymbol{M}_1 = \Omega\boldsymbol{z}$ of the rotating cylinder, and $\boldsymbol{z}$ is the unit vector whose direction coincides with the positive z direction. The constant F_0 is related to the specified difference of the average pressure (3) in the sections S_1 and S_2 by the equality

$$\Delta P = \frac{8\nu^2H}{R_1^2 - R_0^2}\Big(\frac{gH}{2} - F_0\Big), \qquad H = \frac{h}{R_1}, \qquad g = \int\limits_{x_0}^{1} G(x)\,dx.$$

We note that the unknowns u, v, and G are defined by the isolated subsystem (5), (6), which does not contain parameters (except for the specified x_0), and the boundary-value problem (7),(8) subordinate to the subsystem is linear in W, V, and F. This allows the viscous fluid flow (4) to be considered a superposition of the background flow (u, v, G) and the axially homogeneous flow (W, V, and F) induced by the background flow and by the rotation of the inner cylinder.

2. In the absence of the background flow ($u = 0$, $v = 0$, and $G = 0$), the solution of problem (7), (8) coincides with the Couette–Poiseuille solution:

$$\begin{gathered} v_r = 0, \qquad v_\varphi = \frac{\nu}{R_1}\sqrt{\frac{2}{x}}\,V = \frac{\sqrt{2}\,\nu\Omega}{R_1\sqrt{x}}\,\frac{1-x}{1-x_0}, \qquad v_z = -2\,\frac{\nu}{R_1}\,W = \frac{(1-x_0)R_1\Delta P}{4\nu H}\Big(\frac{1-x}{1-x_0} - \frac{\ln x}{\ln x_0}\Big), \\ \Delta P = -\frac{8\nu^2H}{R_1^2 - R_0^2}\,F_0, \qquad Q = \frac{\pi R_1^3\Delta P}{8\nu H}\,(1-x_0^2)\Big(1 + \frac{2}{\ln x_0}\,\frac{1-x_0}{1+x_0}\Big) \end{gathered} \tag{9}$$

(Q is the volumetric fluid flow rate).

This flow regime is characterized by zero radial mass flux: $v_r = 0$. Therefore, the poloidal and azimuthal circulations do not interact. The poloidal circulation is completely determined by the longitudinal pressure gradient, and the azimuthal circulation by the rotation velocity of the inner cylinder. For the same reason, the flow rate, which is proportional to the pressure difference, does not depend on the angular velocity ω.

3. In [3], it was shown that the trivial solution of the nonlinear boundary-value problem (5), (6) is not unique. In the paper cited, the problem of viscous fluid flow in the gap between an outer cylinder at rest and an elongated inner cylinder was studied within the framework of the class of exact solutions (4) ($W = 0$, $V = 0$, and $F = 0$) (the limiting case $R_0 = 0$ was studied in [1]). Investigation of this problem reduces to finding solutions of Eqs. (5) subject to the boundary conditions

$$x = x_0\colon \quad u = v = 0, \quad u' = s, \qquad x = 1\colon \quad u = u' = v = 0, \tag{10}$$

which coincide with (6) for $s = 0$. Figure 1 shows curves of the parameters $u''(x_0, s) = u_0''(s)$, $v'(x_0, s) = v_0'(s)$, and $G(x_0, s) = G_0(s)$ of the Cauchy problem equivalent to problem (5), (10) versus the dimensionless constant s which characterizes the rate of elongation of the inner cylinder. At the point $s = 0$, corresponding to the undeformed cylinder, the values of the listed parameters are different from zero [for $x_0 = 0.01$, $u_0''(0) = -4.10423\cdot 10^3$,

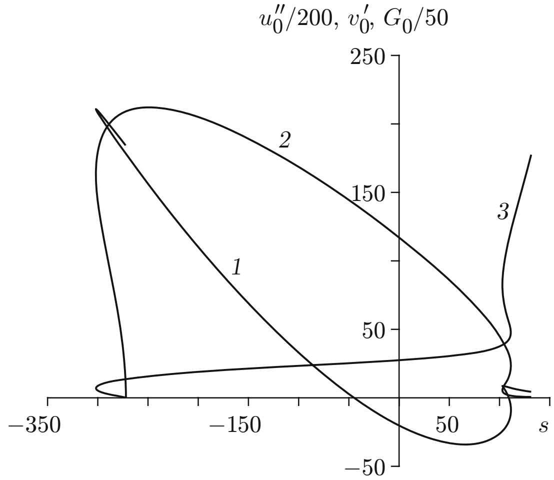

Fig. 1

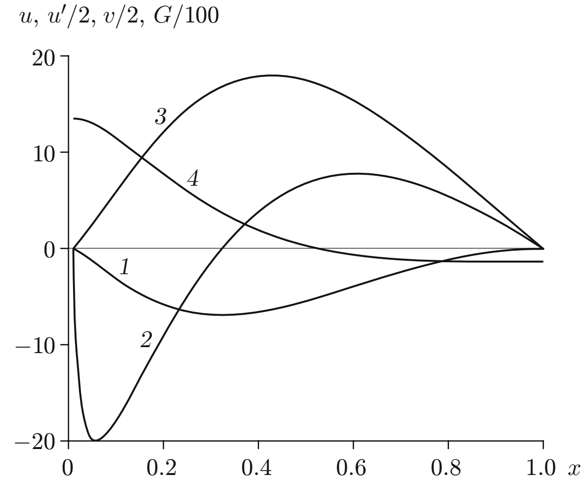

Fig. 2

Fig. 1. Parameters of problem (5), (10): $u_0''(s)$ (1), $v_0'(s)$ (2), and $G_0(s)$ (3).

Fig. 2. Solution of problem (5), (6) for $x_0 = 0.01$: $u(x)$ (1), $u'(x)$ (2), $v(x)$ (3), and $G(x)$ (4).

$v_0'(0) = 116.8048$, and $G_0(0) = 1.35366 \cdot 10^3$]. Thus, for $s = 0$, system (5), (10), and system (5) (6) have a nontrivial solution. The solution of the problem is given in Fig. 2.

Numerical analysis of problem (5), (6) has shown that for all values of the geometrical parameter in the range $x_0 = 0.01$–0.90, the sign of the function v remains unchanged (Fig. 2). This implies that, at all points of the chosen normal section of the gap $Z = \text{const}$, the direction of the dimensionless vector of the axial angular momentum of the background flow $\boldsymbol{M}_2 = vZ\boldsymbol{z}$ does not vary. The sign of the function $u \leq 0$ also remains unchanged (Fig. 2).

The solution obtained describes the rotationally symmetric flow of a viscous fluid in the gap between motionless infinite cylinders with zero flow rate and zero average (throughout the volume) axial angular momentum. The source of this motion is the swirling of the fluids [according to (4)] in the sections $z = h_1$ and $z = -h_1$, which equidistant from the plane S_1. This torsional rotation of the medium leads to the occurrence of radial and longitudinal pressure gradients in it [see the first and third equations in (5)], which induce flows in the corresponding directions.

4. By virtue of the linearity of problem (7), (8), the dependence of its solution on the parameters entering the boundary conditions F_0 and Ω should also be linear. This is easy to see when performing the substitution

$$W = \Omega W_1 - \frac{F_0}{g}\, u', \qquad V = \Omega V_1 - \frac{F_0}{g}\, v, \qquad F = \Omega F_1 + \frac{F_0}{g} \int\limits_x^1 G(t)\, dt, \tag{11}$$

where the new unknowns W_1, V_1, and F_1 satisfy Eqs. (7) with the following boundary conditions not containing parameters:

$$x = x_0\colon \quad W_1 = F_1 = 0, \quad V_1 = 1, \qquad x = 1\colon \quad W_1 = V_1 = F_1 = 0. \tag{12}$$

According to (11), the longitudinal and circumferential velocity components (4) depend linearly on the specified average pressure difference ΔP and the dimensionless axial angular momentum of the rotating cylinder Ω.

Plots of the functions W_1, V_1, and F_1' [at $x_0 = 0.01$, $W_1'(x_0) = -26.286$, $V_1' = -12.658$, and $F'(x_0) = 22.976$] are given in Fig. 3.

An important characteristic of ducted and channeled flows is the flow rate. In view of the boundary conditions and (11), from (4) we obtain

$$Q = \int\limits_{R_0}^{R_1} \int\limits_0^{2\pi} v_z r\, d\varphi\, dr = 2\pi R_1 \Omega q, \qquad q = -\int\limits_{x_0}^1 W_1(x)\, dx. \tag{13}$$

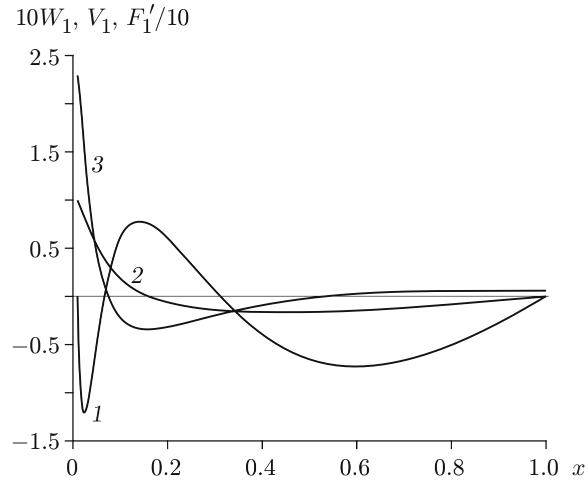

Fig. 3. Solution of problem (7), (12) for $x_0 = 0.01$: $W_1(x)$ (1), $V_1(x)$ (2), and $F_1'(x)$ (3).

In Couette–Poiseuille flow (9), the fluid moves in the direction of decreasing pressure along the channel at a flow rate proportional to $\Delta P/H$. An analysis of formula (13) leads to the different conclusion: the flow rate is proportional to the angular velocity of rotation of the inner cylinder and does not depend on the pressure difference and on the relative length of the gap region H to which this pressure difference is applied.

This unusual flow behavior is due to the presence of background flow of the special form (5), (6) in it, which violates the monotonic longitudinal pressure distribution. This flow, together with the motion induced by the rotation of the inner cylinder, forms a self-coordinate field of inertia forces. To characterize this field, it is reasonable to introduce the vector

$$\boldsymbol{J} = \boldsymbol{z}\,\frac{\partial I}{\partial Z} = \Omega v \boldsymbol{z}$$

with potential I equal to the scalar product of the axial angular momenta of the rotating cylinder and the background flow ($\boldsymbol{M}_1 \cdot \boldsymbol{M}_2 = \Omega v Z$).

It is easy to see that the direction of the mean fluid flow vector $\boldsymbol{j} = Q\boldsymbol{z}$ is unequivocally determined by the field $\boldsymbol{J}$. For this, it is sufficient to notice that system (5)–(7), (12) assumes the transformation of the unknowns u, v, G, W_1, V_1, and F_1 and the corresponding transformation of the vectors:

$$v \to -v: \qquad u \to u, \quad G \to G, \quad W_1 \to -W_1, \quad V_1 \to V_1, \quad F_1 \to -F_1,$$

$$\boldsymbol{J} \to -\boldsymbol{J}: \qquad \boldsymbol{j} \to -\boldsymbol{j}, \quad \boldsymbol{M}_2 \to -\boldsymbol{M}_2.$$

Taking into account the results of numerical investigation of problem (5)–(7), (12) (Fig. 4), according to which, for $v \ge 0$ ($v \le 0$), the constant q is positive (negative) over the entire range of the geometrical parameter x_0, one can conclude that the vectors $\boldsymbol{j}$ and $\boldsymbol{J}$ are codirected. Thus, the average mass transfer in the flow (4) occurs in the direction of increasing potential I (as a function of Z), i.e., to the part of the channel where its inner wall and the background flow rotate in the same direction ($\boldsymbol{M}_1 \cdot \boldsymbol{M}_2 = I > 0$).

5. The streamlines of classical Couette–Poiseuille flow (9) are cylindrical helices with a constant step. Flow of type (4) has much more complex spatial structure. Figure 5 gives qualitative pictures of the isolines of the dimensionless stream functions and the axial angular momentum of the flow (4) determined from the formulas

$$\Psi = \frac{\psi}{\nu R_1} = \Omega \int\limits_{x_0}^{x} W_1(t)\,dt + \Big(Z - \frac{F_0}{g}\Big)u, \qquad M = \frac{r v_\varphi}{\sqrt{2}\,\nu} = \Omega V_1 + \Big(Z - \frac{F_0}{g}\Big)v,$$

where ψ is the true (dimensional) stream function.

The poloidal flow presented in Fig. 5a ($v \ge 0$ and $\Omega > 0$) can be conditionally divided into three regions: the flow-through zone located between the isolines $\Psi = 0$ and $\Psi = Q/(\nu R_1)$ and two recirculation zones: $\Psi < 0$

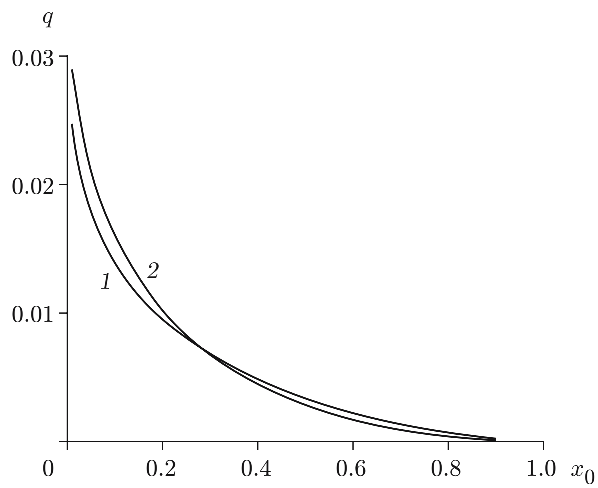

Fig. 4. Dimensionless flow rate versus the ratio of the squared radii of the inner and outer cylinders x_0: 1) $q = Q/(2\pi R_1 \Omega)$; flow described by Eqs. (4) ($v \geq 0$); 2) $q = -Q/(20\pi R_1 F_0)$; Couette–Poiseuille flow (9).

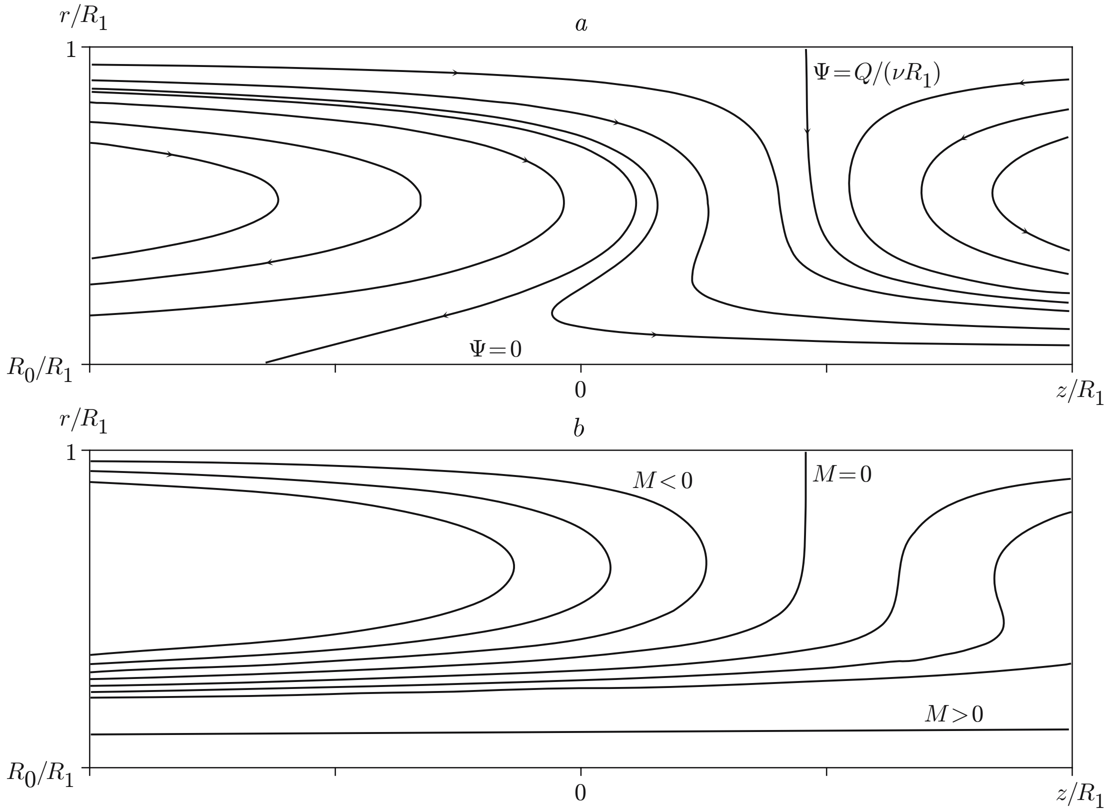

Fig. 5. Isolines of the dimensionless stream functions Ψ (a) and the axial angular momentum M (b).

and $\Psi > Q/(\nu R_1)$. In the flow-through region (for negative values of z), the fluid moves along the outer cylinder (Fig. 5a) and rotates in opposition to the rotation of the inner cylinder (Fig. 5b). In the vicinity of the section $z = 0$, the flow, changing the swirling direction, moves to the inner wall of the gap and flows along it, rotating almost as a rigid body. Since $u \le 0$, the fluid particles move toward the inner cylinder over the entire flow region, i.e., the rotating wall of the gap draws in the fluid.

It should be noted that for $x \approx 1$, the functions G and F' remain almost constant (see Figs. 2 and 3). As a consequence, the pressure at the solid outer wall varies only slightly in the radial direction, suggesting the possible presence of a boundary layer. The collision of the flows of the flow-through zone and the recirculation flow $\Psi > Q/(\nu R_1)$ results in the separation of the boundary layer from the outer solid wall at the point of branching from the last isoline $\Psi = Q/(\nu R_1)$. Flow separation also occurs at the rotating cylinder (isoline $\Psi = 0$), which is accompanied by the occurrence of return flow in the flow-through region (see Fig. 5a).

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